Geometric Formulation of Gauge Theories

W. Drechsler *

Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, München

The development of gauge theories is reviewed beginning with Weyl's theory of 1918 and with the changes introduced by London in the context of quantum mechanics. After a discussion of the Yang-Mills theory and Utiyama's work in the fifties the translation to the modern geometric formulation of gauge theories in terms of fiber bundles is presented.

1. Weyl's Theory of 1918

The notion of gauge invariance was introduced by Hermann Weyl in 1918 in a theory in which he intended to unify the general theory of relativity, describing gravitation, with electromagnetism in using one geometric framework. The idea was to derive the two long range interactions in physics from two aspects of the underlying geometry of space-time: (i) the nonintegrability of the transfer of directions, and (ii) the nonintegrability of the transfer of the scale of lengths. Weyl's basic idea concerning the geometry was that of a manifold which is more general than a pseudo-Riemannian space, i.e. having further properties beyond the metric which vary smoothly from point to point with the global aspects evolving from what he called the "Nahgeometrie" (the geometry in the immediate neighbourhood of a point).

It is well known that the Riemannian geometry used in general relativity is characterized by a non-integrable transfer of directions: Taking by parallel displacement with respect to the Christoffel connection of a metric manifold a vector around a closed loop starting from a point x and returning to x, the direction of the vector has changed compared to the original direction. The amount of change devided by the area of the loop is a measure of the local curvature of the space. Weyl [1, 2] extended this property characterizing Riemannian geometry by asking whether it could also happen that the length – or, to be more precise, the unit of length used in measuring distances – could also change in going from one point of the manifold to a neighbouring one, implying that this unit of length would change in taking it around a closed loop at any point x. Einstein in an addendum to Weyl's paper of 1918 [1] criticised this idea of a pathdependent, i.e. nonintegrable, unit of length by remarking that the existence of sharp spectral lines seen in observing light from different sources in the universe excluded such a dependence of the scale of length on the history of the light ray.

In Weyl's proposal of a nonintegrable transfer of the unit of length – although physically untenable in its original form – the first example of a "gauge theory" appeared on the scene with gauging ("Umeichung") indeed meaning a change in the unit of length in an x-dependent manner on the space-time manifold. Today we do not relate this rescaling of the unit of length and the associated compensating or connection fields (i.e. the gauge potentials) with the electromagnetic interaction as Weyl had proposed. But nevertheless, the concept of a Riemann-Weyl space, W_4, as given by a family of Riemann spaces $(g, \kappa), (g', \kappa'), (g'', \kappa'') \ldots$ has survived as an interesting notion. Let me briefly review it and then go on to describe the modern version of the gauge concept which developed after quantum mechanics and wave mechanics had been established in the years 1925 and 1926, respectively.

Taking two pseudo-Riemannian spaces $(g, \kappa), (g', \kappa')$ in the sequence above, with g denoting the metric and κ the Weyl vector field, there is a *rescaling* of the metric [having Lorentzian signature $(+, -, -, -)$] involved according to

$$g'_{\mu\nu}(x) = e^{\varrho(x)} g_{\mu\nu}(x), \tag{1.1}$$

with the Weyl vector field $\kappa^\mu(x)$ transforming as

$$\kappa'_\mu(x) = \kappa_\mu(x) + \partial_\mu \varrho(x). \tag{1.2}$$

$g_{\mu\nu}(x)$ and $\kappa_\mu(x)$ define a Weyl connection (see below) obeying the so-called semimetric condition [1]

$$D_\varrho\, g_{\mu\nu}(x) = \nabla_\varrho\, g_{\mu\nu}(x) - \kappa_\varrho(x)\, g_{\mu\nu}(x) = 0 \,. \quad (1.3)$$

Here ∇_ϱ denotes the covariant derivative with respect to the Weyl connection with coefficients $\Gamma_{\mu\nu}{}^\varrho$, symmetric in μ and ν, given by

$$\Gamma_{\mu\nu}{}^\varrho = \begin{Bmatrix} \varrho \\ \mu\nu \end{Bmatrix} - \tfrac{1}{2}(\delta_\mu^\varrho\,\kappa_\nu + \delta_\nu^\varrho\,\kappa_\mu - g_{\mu\nu}\,\kappa^\varrho)\,, \quad (1.4)$$

where $\begin{Bmatrix} \varrho \\ \mu\nu \end{Bmatrix}$ are the Christoffel symbols. Correspondingly, the curvature of a Weyl space W_4 is given by the commutator of two Weyl-covariant derivatives of, say, a vector a^ϱ with Weyl weight $w(a)$:

$$(D_\mu D_\nu - D_\nu D_\mu)\,a^\varrho = R_{\mu\nu\sigma}{}^\varrho\,a^\sigma - w(a)\,f_{\mu\nu}\,a^\varrho \quad (1.5)$$

with

$$R_{\mu\nu\sigma}{}^\varrho = \partial_\mu\,\Gamma_{\nu\sigma}{}^\varrho - \partial_\nu\,\Gamma_{\mu\sigma}{}^\varrho + \Gamma_{\mu\lambda}{}^\varrho\,\Gamma_{\nu\sigma}{}^\lambda - \Gamma_{\nu\lambda}{}^\varrho\,\Gamma_{\mu\sigma}{}^\lambda\,, \quad (1.6)$$

and

$$f_{\mu\nu} = \partial_\mu\,\kappa_\nu - \partial_\nu\,\kappa_\mu\,. \quad (1.7)$$

For $\kappa_\mu = 0$ the space W_4 goes over into a Riemannian space-time V_4. The transformation formulae (1.1) and (1.2), representing Weyl's gauge transformation, and (1.7) (with the $f_{\mu\nu}(x)$ being gauge invariant) led Weyl to the identification of the fields $\kappa_\mu(x)$ with the vector potentials $A_\mu(x)$ in electromagnetism and the $f_{\mu\nu}(x)$ with the field strength tensor $F_{\mu\nu}(x)$.

Mathematically, the space W_4 with metric $g_{\mu\nu}$ and Weyl fields κ_μ is characterized by a quadratic fundamental form

$$ds^2 = g_{\mu\nu}(x)\,dx^\mu \otimes dx^\nu\,, \quad (1.8)$$

as well as a linear fundamental form

$$\kappa = \kappa_\mu(x)\,dx^\mu\,. \quad (1.9)$$

Both together determine what Weyl called the "Nahgeometrie", i.e. the infinitesimal geometry of the space-time manifold in the immediate vicinity of a point. Changing the gauge according to (1.1) and (1.2) would multiply (1.8) by a positive number, $e^{\varrho(x)}$, and would change (1.9) by the addition of $d\varrho(x)$.

The linear form (1.9) determines the local relative change of the unit of length l according to

$$\frac{dl}{l} = \kappa_\mu(x)\,dx^\mu\,. \quad (1.10)$$

Integrating this expression along a path C_x from a point x_0 to the point x yields

$$l_x = l_0 \exp\left\{\int_{C_x} \kappa_\mu(x)\,dx^\mu\right\}\,, \quad (1.11)$$

where l_x is the unit of length at x and l_0 is the unit of length at x_0. The integral on the right-hand side of (1.11) is the *real nonintegrable factor* associated with the path C_x expressing Weyl's idea of a nonintegrable transfer of the scale of lengths.

When quantum theory was established in the years 1925–1926, London [3, 4] took up the matter again in 1927 and reformulated Weyl's theory by turning the *real* nonintegrable factor (1.11) into a *complex* nonintegrable phase factor multiplying the quantum mechanical wave function $\psi(x)$, changing thus Weyl's gauge invariance of 1918 to the U(1) gauge invariance as we know it today. Before we come to this point let us, finally, give a quotation from a paper by Weyl who reviewed the development in 1931 [5]. Referring to his 1918 gauge theory he says there [2]: "Alle diese geometrischen Luftsprünge waren verfrüht. Wir kehren zurück auf den Boden der physikalischen Tatsachen." The new "physical facts" clearly were the invention of quantum mechanics and the use of the ψ-function as a representative of matter (i.e. electrons) together with a definite coupling of $\psi(x)$ to the electromagnetic fields which was so successful in the understanding and description of atomic phenomena.

2. U(1) Gauge Invariance in Quantum Theory

London was the first to note that Weyl's real rescaling of the metric in general relativity with the help of an x-dependent element of R_+ (positive real line) had to be replaced by a complex phase transformation of the Schrödinger wave function $\psi(x)$ for electrons in quantum mechanics, i.e. by an x-dependent element [3] of the group U(1)

$$\psi(x) \rightarrow \psi'(x) = e^{-i e \alpha(x)}\,\psi(x)\,, \quad (2.1)$$

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu\,\alpha(x)\,, \quad (2.2)$$

---

[1] Equation (1.3) implies that the tensor $g_{\mu\nu}$ has Weyl weight $+1$ (similarly $g^{\mu\nu}$ has Weyl weight -1). The general Weyl-covariant derivative of a quantity $\phi\dots$ with Weyl weight $w(\phi)$ is

$$D_\varrho\,\phi\dots = \nabla_\varrho\,\phi\dots - w(\phi)\,\kappa_\varrho\,\phi\dots\,.$$

[2] Ref. [5], p. 56, "All these geometric jumps were premature. We return to the ground of physical facts."

[3] e in (2.1) is the electromagnetic charge, with $e = -|e|$ for an electron. We use units for which $\hbar = c = 1$.

such that the probability density $\psi^*(x)\,\psi(x)$ in the non-relativistic Schrödinger theory or the quantity $\bar{\psi}(x)\,\psi(x)$ in Dirac's relativistic theory, with $\bar{\psi}(x) = \psi^\dagger(x)\,\gamma^0$, remains invariant [4]. Since derivatives of the ψ-function appear in the formalism, it is seen with the help of (2.2) that the following generalized derivative of $\psi(x)$ transforms exactly as $\psi(x)$ does:

$$D_\mu\,\psi(x) = (\partial_\mu + i\,e\,A_\mu(x))\,\psi(x) \to D'_\mu\,\psi'(x)$$
$$= e^{-i\,e\,\alpha(x)}\,D_\mu\,\psi(x), \qquad (2.3)$$

where $D'_\mu = \partial_\mu + i\,e\,A'_\mu(x)$ and A'_μ as given in (2.2). The derivative

$$D_\mu = \partial_\mu + i\,e\,A_\mu(x) \qquad (2.4)$$

yields the so-called "minimal electromagnetic coupling". It gives correctly the electromagnetic interaction of nonrelativistic electrons with the electromagnetic fields in the Schrödinger-Pauli theory and, including spin, it yields in the relativistic Dirac theory a correct coupling of relativistic electrons with gyromagnetic ratio $g = 2$ interacting with the fields $F_{\mu\nu}$. The anomalous contribution proportional to $g - 2$ is a small correction (computable in quantum electrodynamics), which would appear in a single particle theory as a so-called non-minimal or Pauli term [5]

$$\tfrac{1}{2}\,(g - 2)\,\mu_B\,\sigma_{\mu\nu}\,F^{\mu\nu}(x)\,. \qquad (2.5)$$

Extending the theory to protons and neutrons, these anomalous magnetic moment terms are sizeable. However, for electrons the minimal electromagnetic replacement

$$\partial_\mu\,\psi(x) \to D_\mu\,\psi(x) = (\partial_\mu + i\,e\,A_\mu(x))\,\psi(x) \qquad (2.6)$$

to be carried out in going over from a free theory to a theory including electromagnetic affects, gives an extremely good description for the interaction of electrons with the electromagnetic fields $F_{\mu\nu}$ in quantum mechanics. Equations (2.1), (2.2) together with (2.6) characterize the abelian $U(1)$ gauge theory with D_μ, defined in (2.4), playing the rôle of a $U(1)$-covariant derivative to be applied to the wave function $\psi(x)$. It is well-known that (2.1) and (2.2) combine the gauge transformations of the first kind in classical electromagnetism

$$A_\mu \to A_\mu + \partial_\mu\,\alpha(x); \quad F_{\mu\nu} \to F_{\mu\nu}\,, \qquad (2.7)$$

where the fieldstrengths, defined by $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$, are gauge *invariant*, with the x-dependent phase transformation (2.1) for the charged matter wave functions in the theory. The probability density $\psi^*(x)\,\psi(x)$ remains *invariant* under gauge transformations (2.1), and Einstein's original criticism of Weyl's theory of 1918 does no longer apply.

One can geometrically visualize the gauge freedom involved in the electromagnetic interaction of a relativistic or nonrelativistic electron by using the language of fiber bundles. Similarly, one can adopt a geometric language for nonabelian gauge theories of the Yang-Mills or a more complicated type, as we shall see in the following sections.

For electromagnetism let us introduce a $U(1)$ fiber bundle by attaching to each space-time point a copy of the unit circle. For easy drawing and visualization let us cut the local circles and stretch them to a line describing the phase angle between 0 and 2π (with the points 0 and 2π being identified). The wave function $\psi(x)$ is then given as a so-called cross section of the $U(1)$ bundle, i.e. as an object possessing a phase which is determined by a smooth map from space-time (being the base space of the bundle) into the bundle space. Changing the section amounts to the gauge transformation (2.1). The formalism must be such that the resulting field equations for the $\psi(x)$ are "covariant", i.e. forminvariant, against such changes of cross sections given by x-dependent transformations from a group – the gauge group – (here the abelian group $U(1)$).

In general, global sections of a principal fiber bundle [6] $P(B, F = G)$ (for which the fiber F is a group G with G acting on itself by right-translation in G) do not exist. On the other hand, local sections always exist by the very definition of a bundle as a generalized space with local product structure of base space and fiber. In this terminology classical electromagnetism in interaction with electrons, which are described in terms of quantum mechanical wave functions (sections), is given by a *trivial* $U(1)$ gauge theory, i.e. by a trivial $U(1)$ bundle. If Dirac monopoles [7] existed in nature, the underlying bundle $P(M_4, G = U(1))$ describing the interactions between electric *and* magnetic charges would be a *nontrivial* fiber bundle requiring several overlapping coordinate patches (two for a

---

[4] We remark in passing that already in 1922 Schrödinger had tried to relate Bohr's quantum conditions to the Weyl theory (compare [3]).

[5] Here μ_B is the Bohr magneton and $\sigma_{\mu\nu} = \tfrac{1}{2}[\gamma_\mu, \gamma_\nu]$, where γ_μ are the Dirac matrices (compare Bjorken and Drell [6]).

[6] See Sect. IV below. For electromagnetism $G = U(1)$, and the base space B of the bundle P would be Minkowski space M_4 (if gravitation is neglected), otherwise it would be V_4 (curved pseudo-Riemannian space-time).

single magnetic monopole). Then the vector potential $A_\mu(x)$, i.e. the connection[7] on $P(M_4, G = U(1))$ in a particular gauge, would be free of singularities in the presence of magnetic poles, and Dirac strings would not show up.

The fact that the basic equations of quantum mechanics for the wave function $\psi(x)$, i.e. the Schrödinger equation in the nonrelativistic case as well as the Dirac equation in the relativistic case, involve the vector potential $A_\mu(x)$ in the form of the $U(1)$-covariant derivative, D_μ, given in (2.4) was extensively discussed by Bohm and Aharonov [8]. This was shown to lead to observable interference phenomena in the diffraction of an electron beam around a domain containing a confined magnetic flux without the electrons ever penetrating into the domain with nonzero field strengths (for recent results in this field compare [9]). This implied the presence of nonlocal effects in quantum mechanics, and the use of path-dependent instead of gauge dependent quantities in the formalism was suggested (compare de Witt [10] and Mandelstam [11]). Finally the question was raised by Wu and Yang [12] what the measurable quantities, actually, are in experiments involving electrons which are described quantum mechanically in terms of a ψ-function (of Schrödinger or four component Dirac type) and the electromagnetic fields. Are the potentials $A_\mu(x)$ themselves measurable? What fixes the gauge if they, indeed, are measurable? Or are the field strengths $F_{\mu\nu}(x)$ the only measurable quantities as in classical electrodynamics. The answer was that it is the path-dependent phase factor (compare (1.11) above) for a path C_{yx} from x to y given by[8]

$$S(C_{yx}) = \exp\left\{-\frac{ie}{\hbar c} \int_{C_{yx}} A_\mu(x') \, dx'^\mu\right\}, \qquad (2.8)$$

which is measurable in electron interference experiments, or – to be more precise – it is the *difference* of various factors $S(C_{yx})$ for different paths C_{yx}, C'_{yx} joining x and y, which is measurable. $S(C_{yx})$, defined in (2.8), is the nonintegrable phase factor associated with the gauge group $U(1)$. It is an element of $U(1)$ associated with the path C_{yx}. Speaking geometrically, (2.8) is the horizontal lift in the bundle space $P(M_4, G = U(1))$ of the curve C_{yx} joining x and y in the base

yielding – in integrating, so to speak, over the gauge history of the electrons in their way from x to y along C_{yx} – the correct phase of the wave function at the point y which determines the interference properties measurable at that point.

The gauge freedom still left in (2.8) is that of gauge transformations at the endpoints x and y of the path C_{yx} according to (compare (2.1))

$$S'(C_{yx}) = S(y) \, S(C_{yx}) \, S^{-1}(x) \qquad (2.9)$$

with

$$S(x) = e^{-ie\alpha(x)} \qquad (2.10)$$

and similarly for $S(y)$ (for a more detailed discussion see [13]). It is apparent from (2.9) and the abelian nature of the gauge group that the nonintegrable phase factor $S(C_{xx})$ associated with a closed loop going through x is gauge *invariant* and hence measurable.

3. Nonabelian Gauge Theories

Electromagnetism is characterized as a gauge theory by the compact abelian group $U(1)$. The question soon arose in theoretical physics after the second world war whether other interactions in nature could be understood in analogy to electromagnetism by considering a gauge theory based on a more complicated compact nonabelian group G. The next essential proposal was to consider the isospin group $G = SU(2)$ and gauge it as it was done by Yang and Mills [14] in 1954. (We heard about the early history of the Yang-Mills theory from Bob Mills before, therefore I shall be brief).

The reasoning in these early days of gauge theory was not geometrical in its character. Not a particular higher dimensional bundle space raised over spacetime yielding connection and curvature forms which obey Cartan's structural equations (see Sect. IV below) marked the beginning of this development. The motivation was physical: Given the isospin symmetry between proton and neutron and describing both particles as an isodoublet (neglecting the mass difference) the direction of the third axis in isospace determines what is a proton and what is a neutron (in the absence of electromagnetism coupling to the charge). Now, in a field theory for nucleons the convention what to call a proton and what a neutron could be made x-dependent. In order to keep track of this x-dependence additional fields – the Yang-Mills fields $B_\varrho(x)$, i.e. the

---

[7] Below we shall systematically translate the various notions appearing in a physically motivated gauge theory defined by a Lagrangian into geometric notions like connection and curvature on the appropriate fiber bundle.

[8] We put back here the factors of $\hbar$ and c.

SU(2)-gauge potentials – had to be introduced and the minimal replacement (2.6) had to be generalized to

$$\partial_\mu \Psi_N(x) \to D_\mu \Psi_N(x) = (\partial_\mu + i B_\mu(x)) \Psi_N(x) . \quad (3.1)$$

Here

$$\Psi_N(x) = \begin{pmatrix} \psi_p(x) \\ \psi_n(x) \end{pmatrix} \quad (3.2)$$

is the nucleon isospinor field, and the Yang-Mills gauge potentials, $B_\mu(x)$, are matrices acting on $\Psi_N(x)$, which read when expanded in terms of the generators of SU(2):

$$B_\mu(x) = B_\mu^i(x) \tau_i , \quad (3.3)$$

where τ_i; $i = 1, 2, 3$, are the Pauli matrices obeying $[\tau_i, \tau_j] = 2 i \varepsilon_{ijk} \tau_k$. (In (3.3) and similarly below a summation over the repeated index i is implied.)

The commutator of two SU(2)-gauge covariant derivatives D_μ and D_ν yields the Yang-Mills field strengths, $B_{\mu\nu}(x)$, being an SU(2)-Lie algebra valued antisymmetrical tensor field given by

$$B_{\mu\nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x) + i [B_\mu(x), B_\nu(x)]$$
$$= B_{\mu\nu}^i(x) \tau_i \quad (3.4)$$

with components $B_{\mu\nu}^i(x)$; $\mu, \nu = 0, 1, 2, 3$; $i = 1, 2, 3$. The $B_{\mu\nu}^i(x)$ can be interpreted as components of a curvature tensor on a principal SU(2) bundle, $P(M_4, G = SU(2))$, over Minkowski space (in a particular gauge), and the Yang-Mills fields $B_\mu(x)$ are the corresponding Lie algebra valued connection fields (in a particular gauge, i.e. pulled back to the base with respect to a local section on $P(M_4, G = SU(2))$ [9].

The fields $B_{\mu\nu}(x)$ depend nonlinearly on the $B_\mu(x)$ in a way similar to the dependence of the curvature tensor on the connection coefficients in a Riemannian space (compare (1.6)). They obey nonlinear field equations due to the fact that the fields $B_\mu(x)$ themselves carry isospin. This is similar to general relativity [10], where the gravitational fields, i.e. the $\Gamma_{\mu\nu}{}^\varrho$, carry energy and momentum (yielding the energy-momentum pseudotensor [15]); and it is in contradistinction to electromagnetism (the *abelian* gauge theory) where the $A_\mu(x)$-field (the photon field) is chargeless.

The Yang-Mills field equations are

$$D^\mu B_{\mu\nu}(x) = \partial^\mu B_{\mu\nu}(x) + i [B^\mu(x), B_{\mu\nu}(x)] = g J_\nu(x), \quad (3.5)$$

$$D_{\{\varrho} B_{\mu\nu\}}(x) = 0 . \quad (3.6)$$

[9] Compare Sect. IV below.
[10] We shall see below that general relativity can be regarded as a gauge theory of the Lorentz group.

The first equality in (3.5) is the definition of the SU(2)-covariant divergence of the $B_{\mu\nu}(x)$ defined in (3.4). The right-hand side of (3.5) represents a Yang-Mills source current [11] given – similar to the Dirac current in QED – by a bilinear expression in terms of the nucleon fields according to [12]

$$J_\nu(x) = J_\nu^i(x) \tau_i \text{ with } J_\nu^i(x) = \bar\Psi_N(x) \gamma_\nu \otimes \tau^i \Psi_N(x) . \quad (3.7)$$

Equations (3.6) are "Bianchi identities" following as integrability conditions from (3.4). Here the curly brackets denote the cyclic sum of the indices $\varrho \mu \nu$. Equations (3.5) and (3.6) are analogous to the field equations in quantum electrodynamics (QED) with the dimensionless coupling constant g replacing the electromagnetic charge e, and with the covariant derivative $D_\mu = \partial_\mu + i [B_\mu,]$ replacing the ordinary derivative ∂_μ in the abelian U(1) theory. As in QED, it follows from the inhomogeneous field equations (3.5) that the isocurrent $J_\nu(x)$ is covariantly conserved, i.e.

$$D^\nu J_\nu(x) = \partial^\nu J_\nu(x) + i [B^\nu(x), J_\nu(x)] = 0 . \quad (3.8)$$

The nucleon field $\Psi_N(x)$ appears – as far as the isospin degrees of freedom are concerned – as a section [13] on a bundle associated to $P(M_4, SU(2))$ with the local fibers being isomorphic to the complex two-dimensional isospinor space, C_2, on which SU(2) acts on the left in the defining 2×2 spinor representation of the group. Changing the local section on $P(M_4, SU(2))$ by performing a transition to another convention regarding the local orientation of the three axis in isospace – which determine what is a proton and what is a neutron at a point x in a whole neighbourhood of this point – yields the transformation

$$B_\mu'(x) = S(x) B_\mu(x) S^{-1}(x) - i S(x) \partial_\mu S^{-1}(x), \quad (3.9)$$

and

$$B_{\mu\nu}'(x) = S(x) B_{\mu\nu}(x) S^{-1}(x), \quad (3.10)$$

where $S(x)$ is an x-dependent element of SU(2) (a Yang-Mills gauge transformation). Observe that the $B_\mu(x)$ transform inhomogeneously as "gauge potentials", i.e. as connection coefficients (see below). Moreover, the field strengths $B_{\mu\nu}(x)$ are no longer gauge

[11] We again introduce a Lie algebra valued current (see (3.7)).
[12] We here assume that $\Psi_N(x)$ is an isodoublet of Dirac spinors.
[13] Associated bundles may possess global sections. The existence of an "isospin" structure on M_4 is assumed here for physical reasons.

invariant, as in the U(1) gauge theory of electromagnetism, but transform *covariantly* under SU(2) gauge transformations.

The gauge transformation for the nucleon isospinor field corresponding to (3.9) and (3.10) reads

$$\Psi'_N(x) = S(x)\,\Psi_N(x)\,. \tag{3.11}$$

As in electromagnetism (compare (2.3)) one again sees that, due to the transformation rule (3.9) for $B_\mu(x)$, the derived field

$$D_\mu\,\Psi_N(x) = (\partial_\mu + i\,B_\mu(x))\,\Psi_N(x) \tag{3.12}$$

has the same transformation behaviour under $S(x)$-transformations as the original field $\Psi_N(x)$, i.e.

$$D'_\mu\,\Psi'_N(x) = S(x)(D_\mu\,\Psi(x))\,, \tag{3.13}$$

where D'_μ is the SU(2)-covariant derivative with respect to $B'_\mu(x)$.

We, finally, write down an SU(2)-gauge covariant Dirac equation for $\Psi_N(x)$:

$$\gamma^\mu(\partial_\mu + i\,B_\mu(x))\,\Psi_N(x) = -i\,m_N\,\Psi_N(x) \tag{3.14}$$

and its adjoint

$$\bar{\Psi}_N(x)\,(\overleftarrow{\partial}_\mu - i\,B_\mu(x))\,\gamma^\mu = i\,m_N\,\bar{\Psi}_N(x)\,, \tag{3.15}$$

where m_N is the nucleon mass. Multiplying (3.14) from the left by $\bar{\Psi}_N(x)\,\tau^i$ and (3.15) from the right by $\tau^i\,\Psi_N(x)$ and adding yields again the SU(2) covariant current conservation (3.8).

The equations (3.5)–(3.7) and (3.14) are the Yang-Mills equations in coupling to a Dirac spinor nucleon field, $\Psi_N(x)$, as material source field with x varying in Minkowski space-time M_4. The twelve fields $B^i_\mu(x)$ appearing in these nonlinear equations are *geometric fields* determining the nonabelian gauge interaction. They are "compensating fields" (see below) keeping track of the change of orientation of the local frames in isospace. Mathematically they are pull backs of a connection on the principal bundle $P(M_4, G = \mathrm{SU}(2))$ with respect to a local section on P (defining a choice of gauge). The physical question what mass corresponds to these geometric fields after quantizing the theory is left open at this point and, in fact, is a problem not yet fully understood even today.

After Yang and Mills had shown how to couple their SU(2)-gauge fields $B_\mu(x)$ to matter fields $\Psi_N(x)$ possessing a nonvanishing isospin, Utiyama [16] presented in 1956 a general Lagrangean formalism and showed how one could introduce into a theory characterized by a Lagrangean $\mathscr{L}^{(0)}$, which is invariant under global (i.e. x-independent) transformations of a group G, a new set of fields (the G-gauge fields) by allowing the transformations of G to become spacetime dependent, and by demanding invariance of an extended Lagrangean $\mathscr{L}$ under these G-gauge transformations. No new geometrical ideas were introduced by Utiyama to motivate or visualize the adopted procedure. The method of gauging a supposedly known theory based on a Lagrangean $\mathscr{L}^{(0)}$ (being, as mentioned, *globally* G-invariant) showed how in a well defined manner for any Group G of order n a set of $4 \cdot n$ new fields – the so-called "compensating" or gauge fields – had to be introduced with a well defined coupling to the original matter fields such that invariance under x-dependent transformations of G is maintained.

Utiyama demonstrated generally that in going over from the Lagrangean $\mathscr{L}^{(0)}(Q^A(x), \partial_\mu Q^A(x))$ depending on the matter fields $Q^A(x)$; $A = 1, 2, \ldots, N$ and their first derivatives transforming under the group G with constant parameters and with the action integral

$$I^{(0)} = \int\limits_\Omega \mathscr{L}^{(0)}\,\mathrm{d}^4 x \tag{3.16}$$

being G-invariant that the interaction Lagrangean of a theory with G as a *gauge* group is obtained by replacing the derivatives ∂_μ in $\mathscr{L}^{(0)}$ by a generalized derivative D_μ constructed with a set of gauge fields $A^a_\mu(x)$ in the following way [14]

$$\tag{3.17}$$

$$\partial_\mu Q^A(x) \rightarrow D_\mu Q^A(x) = \partial_\mu Q^A(x) + A^a_\mu(x)\,[T_a]^A_{\ B}\,Q^B(x).$$

Here T_a is a matrix representation of dimension N of the Lie algebra of the group G, $A^a_\mu(x)$; $\mu = 0, 1, 2, 3$, $a = 1, 2, \ldots, n$ are the "compensating fields" which compensate with their postulated inhomogeneous transformation character under x-dependent transformations of G those terms appearing as a result of the ∂_μ differentiating the x-dependent group action. The postulated G-gauge invariance of the new Lagrangean

$$\mathscr{L}'(Q^A, \partial_\mu Q^A, A^a_\mu) = \mathscr{L}^{(0)}(Q^A, D_\mu Q^A) \tag{3.18}$$

thus led to the extension of the principle of minimal replacement (2.6) in electrodynamics to (3.17), which is now valid for an arbitrary gauge group. Since the gauge fields $A^a_\mu(x)$ introduced by this procedure could play a rôle of their own (i.e. appear disconnected from the material sources), a further term, $\mathscr{L}^{(F)}$, which is

----

[14] Summation over a from 1 to n and over B from 1 to N is implied.

G-gauge invariant in itself, has to be added to yield the Lagrangean $\mathscr{L}$ for the full description of matter and G-gauge fields in interaction.

This is a general scheme allowing the introduction of a new interaction into a theory characterized by a Lagrangean $\mathscr{L}^{(0)}$ which is globally G-invariant. The coupling between the old fields $Q^A(x)$ and the newly introduced gauge fields $A_\mu^a(x)$ is well defined (given by (3.17)), and the field equations and conservation laws follow from the extended action principle $\delta I = 0$ with I given in terms of $\mathscr{L} = \mathscr{L}' + \mathscr{L}^{(F)}$ (compare (3.16)). Utiyama applied this formalism not only to compact groups and internal symmetries but also to the Lorentz group $SO(3, 1)$ and showed that, in a certain sense, contact with Einstein's theory of general relativity could thereby be made. At the end the resulting formulae were then interpreted as applying to a curved space-time manifold although the starting point was a gauging procedure applied to a theory defined over a flat, i.e. Minkowskian, space-time. Although interesting as a heuristic principle applicable locally, I think Utiyama's procedure can, in the case of the space-time symmetries, only be claimed to yield Einstein's theory on a global scale provided additional assumptions are introduced in the formalism.

Although the general procedure described above of introducing a new interaction with a definite coupling into a supposedly known theory was very successful in physics, we like to stress that the underlying geometric structure involved here is again that of a fiber bundle with structural group G raised over flat or curved space-time – or of a Whitney product of such bundles if several gauge interactions charcterized by various groups are present at the same time. The transparency of the physical reasoning in gauge theories is greatly improved by making full use of the geometric language of fiber bundles which has been developed in mathematics during the last sixty years originating from the work of E. Cartan. (For general reference see [17–20] and [13].) We devote the last section to a brief exposé of theese geometric notions.

4. Geometric Approach to Gauge Theories

From the point of view of geometry a gauge theory in physics is given in terms of a principal fiber bundle, $P(B, G)$, over the space-time base B with structural group G. The characteristic property of the bundle $P(B, G)$ is that it is locally a direct product of two

spaces: base space and fiber (for a principal bundle the latter being a group manifold). In $P(B, G)$ the group G acts on itself by right translation ($R_g g' = g' g$; $g, g' \in G$). Stated sloppily, a fiber bundle is a direct product of base space and fiber modulo the action of a group acting on the fiber. (For a general mathematical definition of a fiber bundle see any of the references [17] to [20] or [13]; compare also [21].) The base B in physics may be a M_4 (Minkowski space), a V_4 if general relativistic effects are to be included, a Riemann-Cartan space-time U_4, involving metric and torsion or, finally, a Weyl space W_4 (compare Section I).

Calling $\mathbf{g}$ the Lie algebra of G, a connection Γ is given in terms of a $\mathbf{g}$-valued one-form on $P(B, G)$. A local section $\sigma_i: U_i \to P$ for a neighbourhood $U_i \subset B$, with $\{U_i\}: i \in I$ (I being an indexing set) defining a covering of B, is called a local gauge, and the pull back of ω under σ_i defines a set of $\mathbf{g}$-valued one-forms on the base B (i.e. for $x \in U_i$) denoted by [15]

$$\tilde{\omega}(x) = \sigma_i^* \omega. \tag{4.1}$$

$\tilde{\omega}(x)$ can be expanded on $U_i \subset B$ as

$$\tilde{\omega}(x) = \mathrm{d}x^\mu A_\mu^a(x) T_a, \tag{4.2}$$

where T_a; $a = 1, 2, \ldots, n$ is a basis of $\mathbf{g}$ and $\mathrm{d}x^\mu$ is a natural basis in the local cotangent space $T_x^*(B)$ to B at x. $A_\mu^a(x)$ are the connection coefficients; they are the "gauge potentials" or "compensating fields" of the Lagrangean formulation. Using a Cartan orthonormal moving frame bases in $T_x^*(B)$ given by

$$\theta^i(x) = \lambda_\mu^i(x)\, \mathrm{d}x^\mu, \tag{4.3}$$

where $\lambda_\mu^i(x)$ are the vierbein fields, and similarly for the tangent space $T_x(B)$; i.e. with

$$e_i = \partial_i = \lambda_i^\mu(x)\, \partial_\mu, \tag{4.4}$$

and using a cyclic notation $M_{ab} = -M_{ba}$; $a, b = 1, 2, \ldots, n$ for the generators of G, (4.2) may also be written as [16]

$$\tilde{\omega}(x) = -\theta^i(x)\, \frac{i}{2}\, \tilde{\Gamma}_{iab}(x)\, M^{ab} \tag{4.5}$$

[15] For simplicity we suppress a label $i \in I$ on $\tilde{\omega}(x)$ keeping in mind that $\tilde{\omega}(x)$ is gauge dependent. A gauge on B is given by a collection of local gauges σ_i given in all the U_i of a covering $\{U_i\}$ of B with gauge transformations (4.8) (see below) for x in the intersection of adjacent neighbourhoods.

[16] We use a tilde on $\tilde{\omega}$ and on $\tilde{\Gamma}_{iab}$ to denote the general case. Only if G is identified with the Lorentz group will the tilde be dropped. Furthermore, a factor $-i$ is introduced for convenience (compact generators of G are represented by hermitean matrices).

with connection coefficients $\tilde{\Gamma}_{iab}(x) = -\tilde{\Gamma}_{iba}(x)$ equivalent to the $A_\mu^a(x)$ introduced above. Greek indices are lowered and raised by the metric $g_{\mu\nu}(x)$ and its inverse $g^{\mu\nu}(x)$, respectively.[17] Group indices $a, b, c, \ldots$ are raised and lowered by the Cartan-Killing metric of the group G. The summation convention is adopted for each type of index. For the vierbein fields we have the well-known relations

$$g_{\mu\nu}(x) = \lambda_\mu^i(x)\,\lambda_\nu^k(x)\,\eta_{ik};$$
$$\lambda_\mu^i(x)\,\lambda_j^\mu(x) = \delta_j^i; \quad \lambda_\mu^i(x)\,\lambda_i^\nu(x) = \delta_\mu^\nu. \tag{4.6}$$

Let us, to be definite, assume now that B is a Riemann-Cartan space-time, U_4, with metric and metric compatible torsion (implying that $\nabla_\varrho g_{\mu\nu}(x) = 0$ where ∇_ϱ denotes the covariant derivative with respect to the U_4-connection defined in (4.31) below). We consider the Lorentz frame bundle $P(B, SO(3,1))$ over B with structural group $G = SO(3,1)$. In this case the M^{ab} are denoted by M^{ij}; $i, j = 0, 1, 2, 3$, having Matrix elements

$$[M^{ij}]^k_l = i(\eta^{ik}\delta_l^j - \eta^{jk}\delta_l^i). \tag{4.7}$$

Hence $\tilde{\omega}(x)$, defined by (4.5) and (4.7), is in the Lorentz case a matrix $[\omega(x)]^k_l$ of one-forms, obeying $\omega_{ik}(x) = -\omega_{ki}(x)$ with $\omega_{ik}(x) = \eta_{ij}[\omega(x)]^j_k$, which define a connection on $P(U_4, SO(3,1))$. More exactly, the matrix $[\omega(x)]^j_k$ – called $\omega(x)$ for short – is the pull-back to space-time of a connection in P having the gauge transformation property

$$\omega'(x) = \Lambda(x)\,\omega(x)\,\Lambda^{-1}(x) - \Lambda(x)\,d\Lambda^{-1}(x), \tag{4.8}$$

where $\Lambda(x)$ is an x-dependent element of $SO(3,1)$. The same type of equation as (4.8) applies also for x in the overlap region of two neighbourhoods U_i and U_j on B where the connection is locally given in terms of sections σ_i, σ_j over U_i and U_j, respectively. The transformation $\Lambda(x)$ in (4.8) would then play the rôle of the "transition functions" $g_{ij} \in G$ mediating between two local gauges in the overlap region. As mentioned, we leave out for simplicity any labels on $\omega(x)$ relating to the local coordinate patch and refer to $\omega(x)$ as "the connection" on $P(U_4, SO(3,1))$ remembering the remarks made in connection with (4.1).

Cartan's structural equations for the space-time U_4 are [23, 22] (we suppress the arguments x)

$$\nabla\theta^i = d\theta^i + \omega^i_k \wedge \theta^k = \tau^i, \tag{4.9}$$

$$d\omega_{ij} + \omega_{ik} \wedge \omega_j^k = \Omega_{ij}. \tag{4.10}$$

In (4.9) $\nabla = dx^\mu \nabla_\mu = \theta^i \nabla_i$ denotes the exterior covariant derivative of the fundamental one-forms θ^i with $\wedge$ denoting exterior multiplication. τ^i are the torsion two-forms (Lorentz vector-valued) and $\Omega_{ij} = -\Omega_{ji}$ are the curvature two-forms (Lorentz tensor-valued). Expanding τ^i and Ω_{ij} in a basis of two-forms yields the torsion tensor S_{kl}^i and the U_4-curvature tensor R_{klij} according to

$$\tau^i = \tfrac{1}{2}\theta^k \wedge \theta^l S_{kl}^i, \tag{4.11}$$

$$\Omega_{ij} = \tfrac{1}{2}\theta^k \wedge \theta^l R_{klij}. \tag{4.12}$$

The symmetry relations for the torsion and curvature tensor are

$$S_{ij}^k = -S_{ji}^k; \quad R_{ijkl} = -R_{jikl} = -R_{ijlk}. \tag{4.13}$$

Curvature and torsion are constrained by Bianchi identities following from (4.9) and (4.10) as integrability conditions by exterior differentiation. They read[18]

$$\nabla\tau^i = \Omega_k^i \wedge \theta^k \tag{4.14}$$

and

$$\nabla\Omega_{ij} = 0. \tag{4.15}$$

Equations (4.9) and (4.10) represent a Lorentz gauge covariant set of equations determining the geometry of U_4. The Greek indexed tensorial quantities, for example $S_{\mu\nu}^\varrho$, are obtained in the usual manner by converting the Latin indices with the help of the vierbein fields according to

$$S_{\mu\nu}^\varrho = \lambda_\mu^i \lambda_\nu^j \lambda_k^\varrho S_{ij}^k, \tag{4.16}$$

where the Greek indices refer to a natural (oblique) frame in $T_x(B)$ (for an upper index) or a coframe in $T_x^*(B)$ (for a lower index).

Wave functions or fields representing matter in the theory are represented as sections on bundles *associated* to the respective principal bundle. A Dirac spinor field on a Riemann-Cartan space-time, for example, is determined by the associated bundle[19]

$$\bar{E} = C_4 \times_{\bar{G}} \bar{P}(U_4, \bar{G} = Spin(3,1)), \tag{4.17}$$

where the fiber C_4 is the four-dimensional complex spinor space on which the group $\bar{G}$ acts on the left, with $\bar{G} = Spin(3,1)$ being the universal covering group of $SO(3,1)$ in terms of complex 4×4 matrices. We assume here that a spin structure exists on space-time,

---

[17] Compare [22].
[18] Compare [22] and [24] in this context.
[19] For the notation used here compare Kobayashi and Nomizu [18].

i.e. that the transition to the covering group can be carried out by a map on the whole bundle space P.

The local homomorphism $\text{Spin}(3, 1) \to \text{SO}(3, 1)$ is defined with the help of the Dirac γ-matrices, γ^k, by an x-dependent version of the well-known formula

$$[\Lambda^{-1}(x)]^i_{\ k} \gamma^k = S(\Lambda(x)) \gamma^i S^{-1}(\Lambda(x)), \qquad (4.18)$$

where $S(\Lambda(x))$ is an x-dependent element of $\text{Spin}(3, 1)$, and

$$S^{-1}(\Lambda(x)) = \gamma^0 S^\dagger(\Lambda(x)) \gamma^0. \qquad (4.19)$$

The γ^k (with Latin indices!) are numerically constant 4×4 matrices when a particular representation of the Dirac algebra is adopted [20].

We refer to the bundle $\bar{E}$ as to a bundle over space-time U_4 with fiber C_4 and structural group $\bar{G} = \text{Spin}(3, 1)$ and denote it by

$$\bar{E} = \bar{E}(U_4, F = C_4, \bar{G} = \text{Spin}(3, 1)). \qquad (4.20)$$

A section on $\bar{E}$ is denoted by $\Psi(x)$; $x \in U_4$, having the gauge transformation property

$$\Psi'(x) = S(\Lambda(x)) \Psi(x). \qquad (4.21)$$

The connection on $\bar{E}$ is called the spin connection. It is given by

$$\Gamma(x) = \tfrac{1}{2} \omega_{ij}(x) S^{ij}, \qquad (4.22)$$

where

$$S^{ij} = \frac{i}{4} [\gamma^i, \gamma^j] \qquad (4.23)$$

are the generators of $\text{Spin}(3, 1)$.

The covariant derivative of the section $\Psi(x)$ is

$$D_\mu \Psi(x) = [\partial_\mu + i \Gamma_\mu(x)] \Psi(x)$$
$$= \lambda^i_\mu(x) \left[\partial_i + \frac{i}{2} \Gamma_{ijk}(x) S^{jk} \right] \Psi(x), \quad (4.24)$$

where we have used (4.4) and $\omega_{jk}(x) = \theta^i \Gamma_{ijk}(x)$ in analogy to (4.5).

Now, a wave function or a matter field usually has additional properties beyond spin: it represents charged or neutral particles; it possesses isospin representation character etc. So one has to raise additional fibers over the curved space-time manifold beyond those related to the Lorentz group and consider a local product structure, $B \times \tilde{F}$, where the fiber itself is a direct product of several spaces: $\tilde{F} = F_1 \times F_2 \times \dots \times F_m$. This means that one has to consider a Whitney

product of bundles over B with structural group $\tilde{G} = G_1 \otimes G_2 \otimes \dots \otimes G_m$ of which one factor is $\text{SO}(3, 1)$ if general relativity is included. Correspondingly, the full $\tilde{G}$-covariant derivative for a general wave function $\tilde{\Psi}(x)$, being a section with respect to each of these fiber structures determined by F_i, is given by

$$(4.25)$$
$$\tilde{D}_\mu \Psi(x) = [\partial_\mu + i \Gamma^{(1)}_\mu(x) + i \Gamma^{(2)}_\mu(x) + \dots i \Gamma^{(m)}_\mu(x)] \tilde{\Psi}(x),$$

where the $\Gamma^{(i)}$ are the connections on the corresponding principal bundles $P(B, G_i)$; $i = 1, \dots, m$, in the Whitney product. If, for example, $\Psi_N(x)$ is a charged Dirac spinor isospinor field, the fiber of the associated bundle $\tilde{E}$ to be considered would be

$$\tilde{F} = C_4 \times \text{U}(1) \times C_2, \qquad (4.26)$$

and a section $\Psi_N(x)$ on $\tilde{E}$, transforming under $\tilde{G} = \text{Spin}(3, 1) \times \text{U}(1) \times \text{SU}(2)$, would have the $\tilde{G}$-covariant derivative

$$(4.27)$$
$$\tilde{D}_\mu \Psi_N(x) = [\partial_\mu + i \Gamma_\mu(x) + i e A_\mu(x) \mathbf{1} + i B_\mu(x)] \Psi_N(x)$$

with $\Gamma_\mu(x)$ given by (4.22) and (4.23); $A_\mu(x)$ as described in Sect. II, and $B_\mu(x)$ as defined in (3.3) of Section III.

It is now a simple matter to write down a Dirac-type equation for $\Psi_N(x)$ defined on $\tilde{E}(B = U_4, \tilde{F}, \tilde{G})$. It reads

$$\gamma^\mu(x) \tilde{D}_\mu \Psi_N(x) = -i m_N \Psi(x), \qquad (4.28)$$

with

$$\gamma^\mu(x) = \lambda^\mu_k(x) \gamma^k \qquad (4.29)$$

being an x-dependent set of γ-matrices obeying

$$\{\gamma^\mu(x), \gamma^\nu(x)\} = 2 g^{\mu\nu}(x) \mathbf{1}, \qquad (4.30)$$

where $\{,\}$ denotes the anticommutator, as usual. The $\gamma^\mu(x)$ are covariant constant according to the following relation (compare Schrödinger [26]):

$$(4.31)$$
$$D_\mu \gamma_\varrho(x) = \partial_\mu \gamma_\varrho(x) - \Gamma_{\mu\varrho}{}^\sigma(x) \gamma_\sigma(x) + i [\Gamma_\mu(x), \gamma_\varrho(x)] = 0.$$

Here

$$\Gamma_{\mu\varrho}{}^\sigma = \begin{Bmatrix} \sigma \\ \mu \varrho \end{Bmatrix} + K_{\mu\varrho}{}^\sigma \qquad (4.32)$$

are the U_4-connection coefficients (here again denoted by $\Gamma_{\mu\varrho}{}^\sigma$ – as in Sect. I – where these symbols referred to a Riemann-Weyl space W_4), with

$$K_{\mu\varrho}{}^\sigma = S_{\mu\varrho}{}^\sigma + S^\sigma{}_{\mu\varrho} - S_{\varrho}{}^\sigma{}_\mu. \qquad (4.33)$$

The $S_{\mu\varrho}{}^\sigma$ are defined in (4.16). $\Gamma_\mu(x)$ in (4.31) is the spin connection (4.22).

There are various curvatures appearing now which are related to the nonintegrable transfer of directions

---

[20] See also [25] and Schrödinger [26] in this context.

in the local fibers F_i of $\tilde{E}$: There is (i) the nonintegrable transfer of directions in C_4 (associated to $T_x(B)$) which yields the U_4-curvature composed of Riemannian ($\bar{R}_{ijkl}$) and torsion (P_{ijkl}) contributions [24, 22]

$$R_{ijkl} = \bar{R}_{ijkl} + P_{ijkl}, \qquad (4.34)$$

where $\bar{R}_{ijkl}$ is the usual Riemann-Christoffel tensor (with Latin indices) and

$$P_{ijkl} = \bar{\nabla}_i K_{jkl} - \bar{\nabla}_j K_{ikl} + K_{iks} K_{jl}{}^s - K_{jks} K_{il}{}^s \qquad (4.35)$$

with $\bar{\nabla}_i$ denoting the covariant derivative with respect to the *metric* part of the connection on P (i.e. with respect to the Γ_{ijk} for zero torsion); there is (ii) the nonintegrable transfer of the phase related to the electromagnetic interaction (see (2.8)) with the "U(1)-curvature" $F_{\mu\nu}$ appearing; and, finally, there is (iii) the curvature $B_{\mu\nu}(x)$ for the Yang-Mills or SU(2)-gauge fields. In the present context with the base manifold being a Riemann-Cartan space-time U_4 the derivatives ∂_μ appearing in the definition of the $F_{\mu\nu}$ (see the discussion after (2.7)) and $B_{\mu\nu}(x)$ (see (3.4)) have to be replaced by the covariant derivatives ∇_μ with respect to the $\Gamma_{\mu\varrho}{}^\sigma$ defined in (4.32). In any case, the field strengths $F_{\mu\nu}$ or $B_{\mu\nu}^i$ are to be regarded as components

of a curvature tensor characterizing the geometry of the respective fiber bundles in close analogy to the curvature tensor R_{ijkl} of a connection Γ in Riemannian geometry (compare (1.6)).

One can now go on to investigate various exotic gauge theories by choosing particular groups G_i and fibers F_i in the framework described above, including gravitation from the outset (if necessary extended by the presence of torsion or Weyl degrees of freedom), and investigate the interplay of the various gauge interactions. One can choose, for example, a homogeneous space of a group as fiber [13, 21, 24, 27], or change the Dirac-type Clifford structure in a gauge manner [25, 28], etc. In all the resulting gauge theories the fiber bundle language is, indeed, a very convenient and transparent tool which helps constructing physically relevant models including gravity, and which is indispensable in the topological analysis of global properties of solutions to the nonlinear field equations obtained. Moreover, there is hope that the bundle language may help understanding the problem of the hierarchy of several gauge interactions in nature which are characterized by different strengths.

[1] H. Weyl, Ber. preuss. Akad. Wiss. 1918, p. 465.
[2] H. Weyl, Ann. Phys. **59**, 101 (1919).
[3] F. London, Naturwiss. **15**, 187 (1927).
[4] F. London, Z. Phys. **42**, 375 (1927).
[5] H. Weyl, Naturwiss. **19**, 5 (1931).
[6] J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics, McGraw-Hill Book Company, New York 1964.
[7] P. A. M. Dirac, Proc. Roy. Soc. London **133**, 60 (1931).
[8] D. Bohm and Y. Aharonov, Phys. Rev. **115**, 485 (1959).
[9] Int. Symposium on Foundation of Quantum Mechanics in the Light of New Technology, Tokyo 1983.
[10] B. de Witt, Phys. Rev. **125**, 2189 (1962).
[11] S. Mandelstam, Ann. Phys. (N.Y.) **19**, 1 (1962).
[12] T. T. Wu and C. N. Yang, Phys. Rev. **D 12**, 3845 (1975).
[13] W. Drechsler and M. E. Mayer, Fiber Bundle Techniques in Gauge Theories, Lecture Notes in Physics, vol. 67, Springer-Verlag, Heidelberg 1977.
[14] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).
[15] L. D. Landau and E. M. Lifschiz, Lehrbuch der Theoretischen Physik, vol. II, Feldtheorie, Akademie Verlag, Berlin 1963, p. 327.
[16] R. Utiyama, Phys. Rev. **101**, 1597 (1956).

[17] A. Lichnerowicz, Thèorie globale des connexions et des groupes d'holonomic, Edizioni Cremonese, Roma 1962.
[18] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, vol. I, John Wiley, Interscience, New York 1963.
[19] Y. Choquet-Bruhat, C. de Witt-Morette, and M. Dillard-Bleick, Analysis, Manifolds and Physics, North Holland Publ. Co. Amsterdam 1982.
[20] Ch. Nash and S. Sen, Topology and Geometry for Physicists, Academic Press, London 1983.
[21] W. Drechsler, Fortschr. Phys. **38**, 63 (1990).
[22] W. Drechsler, Fortschr. Phys. **32**, 449 (1984).
[23] E. Cartan, Ann. Ec. Norm. Sup. **40**, 325 (1923); Ann. Ec. Norm. Sup. **41**, 1 (1924).
[24] W. Drechsler, Ann. Inst. H. Poincaré **37**, 155 (1982).
[25] W. Drechsler, Z. Phys. C **41**, 197 (1988).
[26] E. Schrödinger, Sitzungsberichte der Preuss. Akad. Wiss., Phys.-Math. Klasse 1932, p. 436.
[27] W. Drechsler, J. Math. Phys. **26**, 41 (1965); Hadronic J. Suppl. **1**, 193 (1985).
[28] W. Drechsler and W. Thacker, Class. Quantum Grav. **4**, 291 (1987).